SYMMETRY AND MEASUREMENT IN QUANTUM INTERFEROMETRY

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We introduce the foundations for describing multipath quantum interferometry using group theory. The focus here is on linear, lossless interferometry, which exhibits an SU(N) symmetry, but the extension to active interferometry is also discussed. Particle counting is associated with the observables corresponding to operators in the Cartan subalgebra. Interferometric phase is associated with the dual basis to the Cartan weight basis, and a vector phase basis of dimension $N-1$ is associated with N-field quantum interferometry. The vector phase representation allows parametric estimation theory to be employed to calculate lower bounds for determining phase shifts.

Precise interferometric measurements of phase shifts are important for applications such as gravitational wave detection[1] where extremely sensitive phase shifts must be detected, yet complementarity between particle number and phase[2] limits the information which can be extracted from an interferometer. Quantum measurement theory[3,4] applied to interferometry identifies the ultimate quantum limits to measuring phase shifts, but, at present, quantum measurement theory has been applied only to scalar phase measurement, that is, measuring the phase difference between two paths of an interferometer, in the context of one-dimensional harmonic oscillator phase[2] or SU(2) phase, as well as to active, or SU(1,1), interferometry[5]. Our aim here is to study in detail multipath, or SU(N), interferometry, which permits the simultaneous measurement of $N-1$ relative phases and to determine the positive operator-valued measure (POVM), or effect[4], with which to determine the lower bounds on estimating these phase shifts. SU(N) interferometry opens new possibilities for simultaneously measuring multiple phase shifts in quantum interferometry, with implications for improving precision in quantum-limited measurements. Our aim is specifically to (1) determine the SU(N) phase state, (2) identify an important subclass of SU(N) interferometry for which the vector phase POVM can be constructed, (3) relate counting measurements to operators in the Cartan subalgebra of SU(N) and (4) establish the relation between bounds on measuring the vector phase with the Fisher information matrix.

The subject of multipath quantum interferometry is relatively new and includes the results that: an SU(N) interferometer can be realized as some configuration of beam splitters[6]; that the uncertainty in the estimation of a phase shift scales inversely with the number of paths in the interferometer, where the input is coherent light, provided that all N-1 relative phase shifts between contiguous paths are known to be identical[7]; and that the device can be used for measuring the quantum state of light[8]. However, the potential for SU(N) interferometry is much greater than that which has already been explored. Here we identify a class of 'symmetric SU(N) interferometry' for which the POVM corresponding to the measurement of SU(N) vector phase can be determined, and it should be noted that this symmetric N-beam interferometer provides a single-shot measurement of vector phase which is quite distinct from multiple-shot measurements of scalar phase in two-beam interferometry[9].

Lie group theory provides the natural language for describing N-beam interferometers. Yurke *et al*[10] discussed the importance of SU(2) symmetry for passive, linear, lossless interferometry, and Sanders *et al* determined the corresponding scalar phase POVM[5]. Lie group theory can also be used to describe N-beam interferometers for $N > 2$. The unitary operator which transforms N input fields into N output fields can be written as a product of the beam splitter and phase shifter unitary operators[6]. The N fields are associated with annihilation and creation operators $\{(a_k, a_k^\dagger)|k \in \mathcal{Z}_N\}$ satisfying the commutator relations $[a_k, a_l^\dagger] = \delta_l{}^k$. As particle number is conserved at each optical element, it is convenient to introduce the notation, $A_j{}^i \equiv a_j^\dagger a_i$ where $[A_j{}^i, A_l{}^k] = \delta_l{}^i A_j{}^k - \delta_j{}^k A_l{}^i$.

The unitary transformation, corresponding to a passive, linear, lossless optical element (including a beam splitter, a mirror and/or a phase shifter), can thus be expressed as[5,6,10]

$$\mathcal{R}_k^l(\vec{\theta}) = \exp\left\{\imath\vec{\theta}\cdot\left(M_k{}^l, M_l{}^k, M_k{}^k - M_l{}^l\right)\right\} \tag{1}$$

with $M_k{}^k = A_k{}^k$, $M_k{}^l = A_k{}^l + A_l{}^k$ and $M_l{}^k = \imath(A_k{}^l - A_l{}^k)$ for $k < l \le N$. The 50/50 beam splitter corresponds to $\mathcal{R}_k^l(\pi/4, 0, 0)$ and the phase shifter to $\mathcal{R}_k^l(0, 0, \theta)$ (for the mirror $\theta = \pi$). Any SU(N) interferometer unitary transformation can be expressed as a combination of SU(2) elements (1):

$$\mathcal{I}(\Upsilon) = \exp\left(\imath \sum_{k,l=1}^{N} \Upsilon_l{}^k M_k{}^l\right) \tag{2}$$

with real $N \times N$ matrix Υ. Although there are $N-1$ distinct Casimir (invariant) operators, the specification of the particle number sum S is sufficient to determine an irreducible representation as the N-field state consists solely of bosons: only the symmetric irreducible representation appears. For a given irreducible representation (determined by S), we introduce the orthonormal basis $\{|s\vec{m}\rangle\}$ such that

$$S\,|s\vec{m}\rangle = s\,|s\vec{m}\rangle\,,\ \vec{h}\,|s\vec{m}\rangle = \vec{m}\,|s\vec{m}\rangle\,, \tag{3}$$

and the $(N-1)$-dimensional vector $\vec{h}$ is in the Cartan subalgebra.

There is a simple relationship between the basis (3) and the N-field Fock state:

$$|\vec{n}\rangle = \Big|s = \frac{1}{N}\sum_{\nu=1}^{N} n_\nu, \Big\{m_k = \sum_{\nu=1}^{k} n_\nu - k n_{k+1} | k \in \mathcal{Z}_{N-1}\Big\}\Big\rangle. \tag{4}$$

We now have a representation of N-field interferometry as an SU(N) transformation with a bijective mapping between the Fock basis of N fields and the weight basis (3).

We are concerned with a particular class of interferometry corresponding to the symmetric unitary transformation

$$\begin{aligned}\mathcal{I}\left(\vec{\phi},\Upsilon\right) &= \exp\left(\imath\sum\Upsilon_l{}^k M_k{}^l\right)e^{\imath\vec{\phi}\cdot\vec{h}}\exp\left(-\imath\sum\Upsilon_l{}^k M_k{}^l\right)\\ &= \exp\left\{\imath\vec{\phi}\cdot\left[e^{\imath\sum\Upsilon_l{}^k M_k{}^l}\vec{h}e^{-\imath\sum\Upsilon_l{}^k M_k{}^l}\right]\right\};\end{aligned} \tag{5}$$

this class of interferometry is henceforth referred to as 'symmetric SU(N) interferometry'. The quantity $\vec{\phi}$ is an N-dimensional vector phase shift. For example, the unitary operator for the (two-field) balanced Mach-Zehnder interferometer [5,10], is

$$\mathcal{I}\left(\phi,\begin{bmatrix}0 & \pi/4\\ 0 & 0\end{bmatrix}\right) = e^{-\imath(\pi/4)M_1{}^2}e^{-\imath\phi h_1}e^{\imath(\pi/4)M_1{}^2} = \exp\left\{\imath\phi M_2{}^1\right\}. \tag{6}$$

As the SU(N) interferometer can be expressed as a sequence of transformations (1), the general interferometer transformation can be written as a sum over products of the terms above.

Ideal measurement of vector phase for the symmetric interferometer (5) corresponds to the POVM, or effect, $\mathsf{E}_s(\vec{\theta})$[4,5] for fixed s such that (1) $\int d\mathsf{E}_s(\vec{\theta}) = 1$; (2) the spectrum of $\mathsf{E}_s(\theta)$ is strictly positive; and (3) $\mathcal{I}^\dagger(\vec{\phi},\Upsilon)\mathsf{E}_s(\vec{\theta})\mathcal{I}(\vec{\phi},\Upsilon) = \mathsf{E}_s(\vec{\theta}-\vec{\phi})$. If these three criteria are satisfied, then the phase distribution for an input state with density matrix ρ is given by $P(s, \vec{\theta}) = P(s)P_s(\vec{\theta})$ with the conditional distribution $P_s(\vec{\theta})d\mu(\vec{\theta})$ =tr$(\rho d\mathsf{E}_s(\theta))$, where $d\mu(\vec{\theta})$ a measure which normalises the conditional probability distribution, and the conditional phase distribution of the output state is simply the shifted distribution $P_s(\vec{\phi} - \vec{\theta})$.

The appropriate POVM for the symmetric interferometer (5) is, therefore, $d\mathsf{E}_s(\vec{\theta}) = |s\vec{\theta}\rangle_\Upsilon \langle s\vec{\theta}|d\mu(\vec{\theta})$ where

$$\left|s\vec{\theta}\right\rangle_\Upsilon = \mathcal{C}_{\{\vec{m}\}}^{-1/2} \sum_{\{\vec{m}\}} e^{\imath\vec{m}\cdot\vec{\theta}} |s\vec{m}\rangle_\Upsilon \tag{7}$$

is an SU(N) vector phase state, and $\{\vec{m}\}$ is the set of weights for the symmetric representation of SU(N) parametrised by s. The phase state (7) reduces to the rotated SU(2) phase state for $N = 2$[5,11]. An orthonormal basis for the Hilbert space can be constructed with $\mathcal{C}_{\{\vec{m}\}}$ orthonormal phase states. For an arbitrary input state, $|\psi\rangle = \sum_{s\{\vec{m}\}} \psi_{s\vec{m}}|s\vec{m}\rangle$ the conditional phase distribution is given by

$$dP_s(\theta) = \left|\sum_{\{\vec{m}\}} \psi_{s\vec{m}}\, {}_\Upsilon\!\left\langle s\vec{\theta}\middle|s\vec{m}\right\rangle\right|^2 d\mu(\vec{\theta}), \tag{8}$$

and the phase distribution for the output state is $P_s(\vec{\theta}|\vec{\phi}) = P_s(\vec{\theta} - \vec{\phi})$ from which the ultimate limits on estimating the induced vector phase shift $\vec{\phi}$ can be derived. The Fisher information matrix is given by[12]

$$\mathbf{F}_s^{ij} = \int d\mu(\vec{\theta})P_s(\vec{\theta}|\vec{\phi}) \left[\nabla \ln P_s(\vec{\theta}|\vec{\phi})\right] \times \left[\nabla \ln P_s(\vec{\theta}|\vec{\phi})\right]. \tag{9}$$

This covariance matrix is bounded by the requirement that the matrix $\delta\vec{\phi} \times \delta\vec{\phi} - F^{-1}$ is positive definite and $\times$ denotes the outer product of vectors.

In conclusion, we have introduced the principle of SU(N) interferometry for the purpose of measuring vector phase. The POVM for vector phase has been identified for the important special case of 'symmetric SU(N) interferometry'. The link between particle counting measurements and phase estimation[7,13] is established here by noting that the Cartan subalgebra of SU(N) consists linear combinations of particle number operators.

Finally the group theoretical approach developed here can be used to determine the POVM for the symmetric SU(N,M) interferometer which would employ passive, lossless, linear optical elements and two-photon parametric downconverters. The SU(1,1) interferometer has been considered by Yurke *et al*[10] and by Sanders *et al*[5], but the SU(1,1) interferometer can be associated only with a scalar phase. The SU(N,M) interferometer would yield a vector phase measurement, but, unlike for SU(N) interferometry, the Hilbert space is not finite.

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